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THE EFFECT OF A STRIP-SHAPED PUNCH ON A LINEARLY DEFORMABLE FOUNDATION STRENGHTENED BY A THIN COVERING†

N. V. GENERALOVA and Ye. V. KOVALENKO

Moscow

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A method of solving a class of integral equations with a convolution over a finite interval is developed. These equations occur in many mixed problems in the mechanics of continuous media and in mathematical physics. The contact problem in the theory of elasticity of pressure transfer from a punch with a strip-shaped cross-section to a linearly deformable base through a Melan cover reduces to an equation of this kind. The final results are given as analytic formulae which are convenient for specific calculations, with isolated singularities along boundary-condition transition lines.

1. We consider a combined linearly deformable foundation consisting of an elastic layer $|x| < \infty$, $|y| < \infty$, $-H \leqslant z \leqslant 0$ (shear modulus G_2 and Poisson's ratio ν_2), reinforced on its upper surface by a thin elastic covering (G_1, ν_1) of thickness h. The layer either lies without friction on an non-deformable support (case 1) or is attached to it (case 2). Suppose that a rigid punch, with a strip-shaped cross-section and a base described by the function $z = g(x, y)$, is impressed into the boundary $z = h$, $|x| < \infty$, $|y| < \infty$ of this compound medium by a force $P(y)$ per unit length applied with eccentricity b, with the contact domain defined by the inequalities $|x| \leqslant a$, $|y| < \infty$. Frictional forces between the punch and the base are assumed to be zero, and except at the punch surface of the latter is assumed to be unloaded.

We assume that the parameter $n = \theta_1\theta_2^{-1} \gg 1$ $(\theta_j = G_j(1-\nu_j)^{-1}, j = 1, 2)$, but $n\lambda \sim 1$, where $\lambda = ha^{-1} \ll 1$. Then [1] the physical–mechanical properties of the covering can be modelled by a two-dimensional analogue of the Melan cover equations

$$2(\sigma^+ - \sigma^-) + h(\tau_{x,x}^- + \tau_{y,y}^-) = 0$$

$$4G_1 h\Delta(u_{1,x} + \upsilon_{1,y}) = 2(1-\nu_1)(\tau_{x,x}^- + \tau_{y,y}^-) - \nu_1 h\Delta(\sigma^+ + \sigma^-)$$

$$G_1 h\Delta(u_{1,y} - \upsilon_{1,x}) = \tau_{x,y}^- - \tau_{y,x}^-$$

Here Δ is the two-dimensional Laplace operator, u_1 and υ_1 are components of the displacement vector of points on the covering, and $\sigma_z = \sigma^\pm$, $\tau_{xz} = \tau_x^-$, $\tau_{yz} = \tau_y^-$ are given external stresses applied to the lower surface (the minus sign) and upper surface (the plus sign) of the covering.

We will now restrict ourselves to an important special case of the problem posed above. We assume that the functions $g(x, y)$ and $P(y)$ can be decomposed into Fourier series or integrals over y. It is then sufficient to consider $g(x, y) = g(x)e^{-i\beta y}$, $P(y) = Pe^{-i\beta y}$ and perform a superposition of solutions obtained for different values of the parameter β. We will further assume that $c = a|\beta| \leqslant D < \infty$ where $D\lambda \ll 1$ (D = const). The latter inequality is the "condition of applicability of thin-plate theory" [2] and indicates that the external load is smoothly distributed over the surface of the covering $z = h$. The case $\beta = 0$ was investigated in [3], and we will therefore not consider it.

Using the results of [3, 4] we obtain an integral equation for the unknown amplitude $\sigma(x)$ of the contact pressure $\sigma_z(x, y, h) = -\sigma(x)e^{-i\beta y}$ under the punch. Using the notation

$$\xi' = \xi a^{-1}, \quad x' = xa^{-1}, \quad \Lambda = Ha^{-1}, \quad q(x') = (\varepsilon\theta_2)^{-1}\sigma(x)$$

$$f(x') = \delta a^{-1} + \theta x' - g(x)a^{-1}, \quad \varepsilon_j = (1-2\nu_j)[2(1-\nu_j)]^{-1}$$

$$\mu_1 = n\lambda\Lambda^{-1}, \quad \mu_2 = 8\mu_1(1+\kappa_2)^{-2}, \quad \kappa_2 = 3-4\nu_2$$

$$\varepsilon = (1-\varepsilon_2^2)^{-1}, \quad P' = (\varepsilon\theta_2 a)^{-1}P, \quad M = P'ba^{-1}$$

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(where $(\delta + \theta x)e^{-i\beta y}$ is the rigid displacement of the punch under the action of the force $Pe^{-i\beta y}$ and torque $Pbe^{-i\beta y}$ applied to it, and the prime is henceforth omitted), we write it in the form

$$\mathbf{F}q = f \quad (|x| \leqslant 1) \tag{1.1}$$

$$\mathbf{F}q = \int_{-1}^{1} q(\xi) k\left(\frac{\xi - x}{\Lambda}\right) d\xi, \quad k(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K(\alpha) e^{i\alpha t} d\alpha \tag{1.2}$$

where

$$K(\alpha) = \frac{\varepsilon}{u} \times \begin{cases} \dfrac{\Psi_- + \mu_2 u \Gamma_-}{\operatorname{sh} 2u + 2u + 2\mu_1 u \Psi_+} & \text{case 1} \\ \dfrac{2\Gamma_- + 4\mu_2 u(\kappa_2^2 \operatorname{sh}^2 u - u^2)}{2\kappa_2 \Psi_- + (1+\kappa_2)^2 + 4u^2 + 4\mu_1 u \Gamma_+} & \text{case 2} \end{cases} \tag{1.3}$$

$$\Psi_\pm = \operatorname{ch} 2u \pm 1, \quad \Gamma_\pm = \kappa_2 \operatorname{sh} 2u \pm 2u, \quad u = \sqrt{\alpha^2 + \eta^2}, \quad \eta = c\Lambda$$

If we reformulate the problem so that the base is taken to be an elastic half-space, we still have the integral equations (1.1) and (1.2), but with $\Lambda = 2\lambda n$ and

$$K(\alpha) = \frac{u + \varepsilon}{u(u+1)} \tag{1.4}$$

Equation (1.1), (1.2) has to be supplemented with the statics conditions

$$P = \int_{-1}^{1} q(x) dx, \quad M = \int_{-1}^{1} x q(x) dx \tag{1.5}$$

which express the equilibrium of the punch at the base.

Note that the structure of the solution of integral equation (1.1), (1.2) and its properties are mainly governed by the behaviour of the kernel symbol (1.3), (1.4) along the real axis. Below we assume that $K(\alpha)$ is a real, even, continuous and positive function for $|\alpha| < \infty$, with the following asymptotic behaviour

$$K(\alpha) = A + B\alpha^2 + O(\alpha^4) \quad (\alpha \to 0) \tag{1.6}$$

$$K(\alpha) = |\alpha|^{-1}[1 + C|\alpha|^{-1} + O(\alpha^{-2})] \quad (|\alpha| \to \infty)$$

(There is an inaccuracy in the asymptotic formulae (1.7) in [3]. They should be replaced by the expressions (1.6) given here.)

This is the framework for the contact problem in question for the transmission of pressure to a linearly deformable medium through a covering. The symbols $K(\alpha)$ of the form (1.3) and (1.4) satisfy conditions (1.6) in which, for example, for case (1.4) one must put

$$A = \frac{\varepsilon + \eta}{\eta(1+\eta)}, \quad B = -\frac{1}{2}\frac{\varepsilon + 2\eta\varepsilon + \eta^2}{\eta^3(1+\eta)^2}, \quad C = \varepsilon - 1 \tag{1.7}$$

On the basis of (1.6) it has been proved [5] that if $f(x) \in H_1^r(-1, 1)$ $(0 < r \leqslant 1)$, the integral equation (1.1), (1.2) is uniquely solvable in the space $L_p(-1, 1)$ of functions integrable in the interval $[-1, 1]$ with degree p $(1 < p < 2)$, and its solution $q(x)$ can be represented in the form

$$q(x) = (1 - x^2)^{-1/2} \omega(x) \tag{1.8}$$

$$\omega(x) \in H_0^s(-1, 1), \quad s = \inf(r, (p-1)/p)$$

We also have the well-posedness relations

$$\|q\|_{L_p} \leqslant D_1(\Lambda) \|f\|_{H_1^r}, \quad \|\omega\|_{H_0^s} \leqslant D_2(\Lambda) \|f\|_{H_1^r} \tag{1.9}$$

Here $D_1(\Lambda)$ and $D_2(\Lambda)$ are constants bounded for any fixed $\Lambda \in (0, \infty)$, and $H_m^\gamma(-1, 1)$ is the space of functions whose mth derivatives satisfy the Hölder condition with index γ when $|x| \leqslant 1$.

2. To construct the solution of (1.1), (1.2) with kernel symbol (1.6) we use a modification of the Galerkin projection method [6, 7]. We introduce the Hilbert spaces $H_{-1/2}(-1, 1)$ and $L_2(-1, 1)$ and specify within them two complete systems of coordinate (basis) functions

$$\{\varphi_m(x)\},\ \{\psi_m(x)\}\ \ (m=0,1,2...) \tag{2.1}$$

whose linear envelopes are a limitingly dense sequence of subspaces.

The space $H_{-1/2}(-1, 1)$ is defined in [5, 7] as the closure of $L_p(-1, 1)$ $(1 < p < 2)$ in the norm

$$\|\varphi\|^2_{H_{-1/2}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} K(\alpha)|\Phi(\alpha)|^2 d\alpha,\ \ \Phi(\alpha) = \int_{-1}^{1}\varphi(\xi)e^{i\alpha\xi/\Lambda}d\xi$$

According to (1.8) the solution $q(x)$ has a root singularity at the points $x = \pm 1$ and so in the first case one has to use a system of functions with singularities of this type. This is not the case for the second system because the right-hand side $f(x)$ of Eq. (1.1) and the residue $z_N(x)$ (see below) are smooth functions.

We represent an approximate solution in the form

$$q_N(x) = \sum_{m=0}^{N} a_m\varphi_m(x) \tag{2.2}$$

where the coefficient a_m is determined from the orthogonality of $z_N = \mathbf{F}q_N - f$ to elements of the second coordinate sequence

$$(z_N, \psi_j)_{L_2} = 0\ \ (j=0,1,2,...,N) \tag{2.3}$$

Relations (2.3) reduce to a system of linear algebraic equations of $(N + 1)$th order in the unknowns a_m

$$\sum_{m=0}^{N} c_{jm}a_m = b_j\ \ (j=0,1,2,...,N) \tag{2.4}$$

$$c_{jm} = (\mathbf{F}\varphi_m, \psi_j)_{L_2},\ \ b_j = (f, \psi_j)_{L_2}$$

which can be shown [7] to have a non-degenerate matrix because of the well-posed solvability of the integral equation (1.1), (1.2), (1.6) in $L_p(-1, 1)$ $(1 < p < 2)$. Moreover, by (1.9) and the previously described properties of the coordinate functions, we can write

$$\|q - q_N\|_{L_p} \leqslant C_1(\Lambda)\|q - q_N\|_{H_{-1/2}} \leqslant C_2(\Lambda)\|z_N\|_{H_1^\gamma} \leqslant C_3(\Lambda)\|z_N\|_{L_2} \to 0\ \ (N \to \infty)$$

(where $C_1(\Lambda)$, $C_2(\Lambda)$, $C_3(\Lambda)$ are constants) which demonstrates the applicability of Galerkin's method for solving the original integral equation for all values of the parameter $\Lambda \in (0, \infty)$.

We take the basis functions to be $(m = 0, 1, 2, \ldots)$

$$\left\{\frac{T_m(x)}{\sqrt{1-x^2}}\right\},\ \{P_m^*(x)\},\ \ P_m^*(x) = \sqrt{\frac{2m+1}{2}}P_m(x)$$

where $T_m(x)$ and $P_m(x)$ are Chebyshev and Legendre polynomials. Using the definite integrals

$$\int_{-1}^{1}\frac{T_m(x)}{\sqrt{1-x^2}}e^{\pm i\alpha x}dx = \pi e^{\pm\pi m i/2}J_m(\alpha)$$

$$\int_{-1}^{1}P_m(x)e^{\pm i\alpha x}dx = \sqrt{\frac{2\pi}{\alpha}}e^{\pm\pi m i/2}J_{m+1/2}(\alpha)$$

(where $J_\nu(x)$ is the Bessel function) we represent the matrix elements of system (2.4) in the form

$$c_{jm}=\sqrt{\pi\Lambda(2j+1)}\cos\frac{\pi(m-j)}{2}\int_0^\infty\frac{K(\alpha)}{\sqrt{\alpha}}\zeta_{jm}\left(\frac{\alpha}{\Lambda}\right)d\alpha$$
$$\zeta_{jm}(x)=J_{j+1/2}(x)J_m(x) \tag{2.5}$$

If we take the splines

$$\varphi_m(x)=\frac{\psi_m(x)}{\sqrt{1-x^2}},\quad \psi_m(x)=\psi\left(\frac{x-x_m}{r}\right)\quad (m=0,1,2,\dots,N)$$
$$\psi(x)\begin{cases}1-|x| & (|x|<1)\\ 0 & (|x|>1)\end{cases} \tag{2.6}$$

to be the coordinate functions, we arrive at the variational-difference method [7] for solving integral equation (1.1), (1.2), (1.6). Here the nodes $x_m = -1 + r(2m + 1)/2$ $(m = 0, 1, 2, \dots, N)$ cover the interval $[-1, 1]$ with steps of $r = 2(N + 1)^{-1}$.

Introducing the functions (2.6) into the second and third formulae of (2.4) and using the quadrature approximation [8]

$$\int_{-1}^{1}\frac{\psi(x)e^{-i\alpha x}dx}{\sqrt{|1-(rx+x_m)^2|}}\approx\frac{2}{\sqrt{1-x_m^2}}\int_0^1\psi(x)\cos\alpha x\,dx=\frac{2(1-\cos\alpha)}{\alpha^2\sqrt{1-x_m^2}}$$

as $r\to 0$, we have

$$c_{jm}=c_{mj}=\frac{4r\Lambda}{\pi\sqrt{1-x_m^2}}\int_0^\infty K\left(\frac{\Lambda}{r}\alpha\right)\frac{(1-\cos\alpha)^2}{\alpha^4}\cos\left(\alpha\frac{x_m-x_j}{r}\right)d\alpha$$
$$b_j=\int_{x_j-r}^{x_j+r}f(x)\psi\left(\frac{x-x_j}{r}\right)dx\approx rf(x_j) \tag{2.7}$$

In the last relation in (2.7) we have used the δ-shape of the system of coordinate functions $\psi_m(x)$.

The key feature of the version of Galerkin's projection method being used here is the evaluation of the integrals of rapidly oscillating functions in (2.5) and (2.7). This difficulty is overcome by an algorithm from [9]. As an example, for the quadrature in formula (2.5) this algorithm gives

$$\int_0^\infty\frac{K(\alpha)}{\sqrt{\alpha}}\zeta_{jm}\left(\frac{\alpha}{\Lambda}\right)d\alpha=\sum_{n=0}^\infty\sqrt{\alpha_n}K(\alpha_n)\left[S_{jm}\left(\frac{e_n}{\Lambda}\right)-S_{jm}\left(\frac{c_n}{\Lambda}\right)\right]$$
$$c_n=nl,\quad e_n=(1+n)l,\quad \alpha_n=(c_n+e_n)/2$$
$$S_{jm}(x)=-\frac{x[\zeta_{j,m+1}(x)-\zeta_{j+1,m}(x)]}{m^2-(j+1/2)^2}+\frac{\zeta_{jm}(x)}{m+j+1/2}$$

Here l is a sufficiently small number which describes the distance between the nodes c_n and e_n.

Having constructed the function $q_N(x)$ in accordance with (2.2), we then find the amplitudes of the force P and torque M applied to the punch. In the first and second system of coordinate functions they, respectively, acquire the forms

$$P=\pi a_0,\quad r\sum_{j=0}^N\frac{a_j}{\sqrt{1-x_j^2}};\quad M=\frac{\pi}{2}a_1,\quad r\sum_{j=0}^N\frac{x_ja_j}{\sqrt{1-x_j^2}}$$

We noted above that the proposed Galerkin's method works for all values of the parameter $\Lambda\in(0,\infty)$. However, a numerical analysis of specific problems shows that when $\Lambda\to 0$ its efficiency falls because of the substantial increase in the number of equations in system (2.4) used to obtain the approximate solution (2.2) to a specified accuracy. We therefore present an algorithm for investigating the integral equation (1.1), (1.2), (1.6) which can be sensibly used for small values of Λ.

We apply a well-known result [10] of Krein, and restrict ourselves to the case $f(x) \equiv f = \text{const}$. Then the leading term of the asymptotic solution of the original integral equation can be represented in the form [5]

$$q(x)=\frac{f}{\Lambda}\left[\chi\left(\frac{1+x}{\Lambda}\right)+\chi\left(\frac{1-x}{\Lambda}\right)-\upsilon\left(\frac{x}{\Lambda}\right)\right] \tag{2.8}$$

where the functions $\chi(t)$ and $\upsilon(t)$ are determined by the integral equations

$$\int_0^\infty \chi(\tau)k(\tau-t)d\tau=1 \quad (0\leqslant t<\infty) \tag{2.9}$$

$$\int_{-\infty}^\infty \upsilon(\tau)k(\tau-t)d\tau=1 \quad (|t|<\infty) \tag{2.10}$$

The solution of Eq. (2.10) is constructed using the convolution theorem for Fourier transformations and has the form

$$\upsilon(x\Lambda^{-1})=A^{-1} \tag{2.11}$$

The solution of Eq. (2.9) can be obtained by a Wiener–Hopf method whose use requires the kernel symbol $K(\alpha)$ to be approximated by an expression

$$K_*(\alpha)=\frac{\sqrt{\alpha^2+h_3^2}}{\alpha^2+h_4^2}\exp\left(\frac{h_1}{\sqrt{\alpha^2+h_2^2}}\right), \quad K_*(0)=A \tag{2.12}$$

that is consistent with formulae (1.6).

Basing ourselves on the results in [5] we write

$$\chi(t)=\frac{1}{K_-(0)}\frac{1}{2\pi i}\int_\Gamma \frac{e^{-i\zeta t}d\zeta}{K_+(\zeta)\zeta} \tag{2.13}$$

In (2.13) the contour Γ is a line lying just above the real axis in the complex $\zeta = \alpha + i\sigma$ plane, and

$$K_*(\zeta)=K_+(\zeta)K_-(\zeta)$$

where $K_+(\zeta)$ and $K_-(\zeta)$ are regular in the upper half-plane $\operatorname{Im}\zeta > -\sigma_+$ and lower half-plane $\operatorname{Im}\zeta < \sigma_-$ and have no zeros in these half-planes.

Because the technique for factorizing functions of the form (2.12) is well known [3, 11] we will merely give the final results

$$K_\pm(\zeta)=\frac{\sqrt{\zeta\pm ih_3}}{\zeta\pm ih_4}e^{h_1\mu_\pm(\zeta)}, \quad \mu_\pm(\zeta)=\frac{\pm i}{\pi\sqrt{\zeta^2+h_2^2}}\ln\frac{\zeta+\sqrt{\zeta^2+h_2^2}}{\pm ih_2} \tag{2.14}$$

From (2.12) and (2.14) it follows that

$$K_-(0)=\overline{K_+(0)}=\sqrt{Ai} \tag{2.15}$$

We substitute (2.14) and (2.15) into (2.13) and for convenience convert the latter from an integral Fourier transform into a Laplace–Carson transform putting $\zeta = ip$. We obtain

$$\begin{gathered}\chi(t)=\frac{1}{\sqrt{A}}\frac{1}{2\pi i}\int_L \frac{X(p)}{p}e^{pt}dp \\ X(p)=\frac{p+h_4}{\sqrt{p+h_3}}e^{-h_1\mu(p)}, \quad \mu(p)=\frac{1}{\sqrt{p^2-h_2^2}}\ln\frac{p+\sqrt{p^2-h_2^2}}{h_2}\end{gathered} \tag{2.16}$$

Here L is a line lying just to the right of the imaginary axis in the complex p plane.

Then, in order to obtain results that are applicable in practice we approximate the exponent in the second formula in (2.16) by the expression [11]

$$\exp[-h_1\mu(p)] \approx 1 - h_1\mu(p) \tag{2.17}$$

Note that the error in approximation (2.17) when $\operatorname{Re} p > 0$, $\operatorname{Im} p = 0$ does not exceed 1% for the values of the constants h_1 and h_2 found below.

Substituting (2.17) into (2.16) and using the table in [12], we write

$$\begin{aligned}
&\chi(t) = \frac{1}{\sqrt{A}}\left[\frac{e^{-h_3 t}}{\sqrt{\pi t}} + \frac{h_4}{\sqrt{h_3}}\operatorname{erf}\sqrt{h_3 t} + I_1(t)\right] \\
&I_1(t) = -\frac{h_1}{\pi\sqrt{h_3}}\int_0^t \operatorname{erf}\sqrt{h_3(t-\tau)}\,R_1(\tau)\,d\tau \\
&R_1(t) = K_0(h_2 t) + h_4\left[\frac{\pi}{2h_2} - \int_t^\infty K_0(h_2 s)\,ds\right]
\end{aligned} \tag{2.18}$$

where $K_0(x)$ is the Macdonald function.

We determine the integral characteristic of the solution P constructed by substituting (2.8) into (1.5). Bearing in mind relations (2.11) and (2.18), we have

$$P = 2f\int_0^\kappa \chi(\tau)\,d\tau - \frac{f\kappa}{A} = \frac{2f}{\sqrt{Ah_3}}[\operatorname{erf}\sqrt{h_3\kappa} + h_4 R_2(\kappa) + I_2(\kappa)] - \frac{f\kappa}{A}$$

$$\kappa = \frac{1}{\Lambda},\quad R_2(t) = \frac{2h_3 t - 1}{2h_3}\operatorname{erf}\sqrt{h_3 t} + \sqrt{\frac{t}{\pi h_3}}\,e^{-h_3 t}$$

$$I_2(t) = -\frac{h_1}{\pi}\int_0^t R_2(t-\tau)R_1(\tau)\,d\tau$$

3. As an example we present numerical calculations for problem (1.1), (1.2), (1.4) by the three methods given in Section 2, for $f(x) = f = \text{const}$ and various values of the dimensionless parameter $\Lambda \in (0, \infty)$. Suppose that in (1.4) $\nu_2 = 0.3$ ($\varepsilon = 1.08889$), $c = 1$. We select the constants h_j ($j = 1, 2, 3, 4$) in (2.12) using Eq. (1.7), relations

$$A = \frac{h_3}{h_4^2}\exp\left(\frac{h_1}{h_2}\right),\quad h_3\left(\frac{B}{A} + \frac{h_1}{2h_2^3}\right) = \frac{1}{2h_3} - A\exp\left(-\frac{h_1}{h_2}\right),\quad C = h_1$$

and the requirement to minimize $|1 - K_*(\alpha)K^{-1}(\alpha)|$ for $|\alpha| < \infty$.

The results of the calculations are given below ($h_1 = 0.08889$ for all values of Λ)

Λ	0.25	0.5	1	2
h_2	2.1413	2.4137	0.8193	18007
h_3	0.2024	0.4237	0.9321	1.9478
h_4	0.2219	0.4555	0.9974	1.9937

Note that the relative error of approximation (2.12) for the given values of h_j does not exceed 0.25% over the entire real axis. Then the error of the approximate solution (2.8), (2.11) and (2.18) using (2.17) amounts to about 1%.

The values of the dimensionless contact pressure amplitude $\varepsilon q(x)f^{-1}$ and the amplitude of the punch embedding force εPf^{-1} are shown in Table 1. The rows are grouped in triplets, in each of which the first row shows the results of the calculations using the first algorithm, etc.

We can draw the following conclusions from the data in Table 1:

1. the first and second methods can be sensibly used for $0.5 \leqslant \Lambda < \infty$; for such values of Λ the number of equations in system (2.4) does not exceed 5 and 20 respectively, and the relative errors of the results are less than 2%;
2. the small Λ method works when $\Lambda \leqslant 2$;
3. matching of the results is observed when $0.5 \leqslant \Lambda \leqslant 2$;
4. if the effect of the reinforcing cover is ignored in the case $\Lambda = 1$, comparison with results from [5, 13] shows that the error exceeds 14%.

Table 1

Λ	$x=0$	0.2	0.4	0.6	0.8	0.9	0.95	$\varepsilon P f^{-1}$
0	1.101	1.112	1.151	1.245	1.521	1.974	2.665	2.994
[5, 13]								
	1.097	1.110	1.147	1.253	1.527	1.966	2.658	2.983
0.25	1.109	1.123	1.159	1.251	1.525	1.988	2.684	3.016
	1.166	1.179	1.219	1.318	1.608	2.086	2.813	3.164
	1.163	1.176	1.219	1.315	1.605	2.089	2.815	3.175
0.5	1.165	1.175	1.215	1.317	1.609	2.091	2.813	3.173
	1.175	1.185	1.226	1.326	1.618	2.098	2.826	3.184
	1.246	1.265	1.305	1.409	1.685	2.170	2.899	3.337
1	1.250	1.268	1.309	1.407	1.694	2.173	2.905	3.342
	1.252	1.264	1.303	1.401	1.687	2.166	2.894	3.335
	1.210	1.221	1.269	1.375	1.680	2.159	2.870	3.286
2	1.213	1.225	1.271	1.373	1.678	2.160	2.873	3.292
	1.203	1.215	1.257	1.360	1.661	2.156	2.865	3.271
4	1.203	1.213	1.261	1.366	1.668	2.149	2.858	3.269
	1.207	1.215	1.260	1.365	1.669	2.150	2.861	3.270
8	1.189	1.201	1.243	1.347	1.649	2.130	2.839	3.235
	1.191	1.203	1.242	1.347	1.645	2.135	2.842	3.239

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